Necessary conditions for Hölder regularity gain of $\overline{\partial}$ equation in $\mathbb{C}^3$

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Abstract

Suppose that a smooth holomorphic curve V has order of contact η at a point w_0 in the boundary of a pseudoconvex domain Ω in $\mathbb{C}^3$. We show that the maximal gain in Hölder regularity for solutions of the $\bar{\partial}$-equation is at most $\frac{1}{\eta}$.

1 Introduction

Let Ω be a given domain in $\mathbb{C}^n$ and α be a $\bar{\partial}$-closed form of type $(0,1)$ in Ω. The $\bar{\partial}$-problem consists of finding a solution u of $\bar{\partial}u = \alpha$ that satisfies certain boundary regularity estimates as measured by either L^2 or L^p norms or in Hölder norms.

When Ω is strongly pseudoconvex, in the L^2-sense, Kohn [5, 7, 8] showed that for any $s \geq 0$, there is a canonical solution of $\bar{\partial}u = \alpha$ such that

$$|||u|||_{s+\epsilon} \leq C \|\alpha\|_s \quad \text{and} \quad u \perp A(\Omega) \cap L^2(\Omega), \tag{1.1}$$

with $\epsilon = \frac{1}{2}$. (We say u is the canonical solution if $u \perp A(\Omega) \cap L^2(\Omega)$.) Here, $\|\cdot\|_s^2$ is the L^2-Sobolev norm of order s and the norm $|||\cdot|||_{s+\epsilon}$ measures tangential derivatives near the boundary of order $s+\epsilon$ in the tangential directions. Kohn showed that if U satisfies $\Box U = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})U = \alpha$, and if $\bar{\partial}\alpha = 0$, then $u = \bar{\partial}^* U$ is the canonical solution of $\bar{\partial}u = \alpha$. To prove regularity for this solution, Kohn proved the a priori estimate

$$|||\phi|||_\epsilon^2 \leq C(\|\bar{\partial}\phi\|^2 + \|\bar{\partial}^*\phi\|^2 + \|\phi\|^2) \tag{1.2}$$

with $\epsilon = \frac{1}{2}$. Here, $\phi \in C^\infty_{(0,1)}(W) \cap \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$ is compactly supported in the neighborhood W of the boundary point w_0. Using this estimate and a bootstrap argument, Kohn proved (1.1). Stein and Greiner [6] later extended (1.1) to similar estimates in L^p and Hölder spaces. For example, if $\|\cdot\|_{\Lambda^s(\Omega)}$ is the Hölder norm of degree s, then Stein and Greiner proved that u satisfies

$$\|u\|_{\Lambda^{s+\epsilon}(\Omega)} \leq C \|\alpha\|_{\Lambda^s(\Omega)}, \tag{1.3}$$

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with $\epsilon = \frac{1}{2}$.

Kohn extended his L^2 results to when Ω is a regular finite 1-type pseudoconvex domain in $\mathbb{C}^2$. To define a regular finite 1-type, we measure the order of contact of a given holomorphic curve at $w_0 \in b\Omega$. Let V be a one-dimensional *smooth* variety parametrized by $\zeta \to \gamma(\zeta) = (\gamma_1(\zeta), \cdots, \gamma_n(\zeta))$, where $\gamma(0) = w_0$ and $\gamma'(0) \neq 0$. We define the order of contact of the curve by $\nu_o(R \circ \gamma)$, where R is a defining function of Ω and $\nu_o(g)$ is just the order of vanishing (an integer at least equal to 2) of g at 0. We then define the type, $T^{reg}_\Omega(w_0) = \sup\{\nu_o(R \circ \gamma); \text{all } \gamma \text{ with } \gamma(0) = w_0, \gamma'(0) \neq 0\}$. Further, we can define the regular type of Ω by $T^{reg}(\Omega) = \sup\{T^{reg}_\Omega(w_0); w_0 \in b\Omega\}$. Kohn [9] proved that if Ω is a regular finite 1-type pseudoconvex domain in $\mathbb{C}^2$, then (1.1) holds for $\epsilon = \frac{1}{T^{reg}(\Omega)}$. Similarly, Nagel-Rosay-Stein-Wainger [13] showed that (1.3) also holds for the same ϵ.

In order to discuss similar estimates in $\mathbb{C}^n$, it is important to consider the order of contact of *singular curves*. We define the order of contact of a holomorphic curve parametrized by $\zeta \to \gamma(\zeta)$, with $\gamma(0) = w_0$, by $C_\Omega(\gamma, w_0) = \frac{\nu_o(R\circ\gamma)}{\nu_o(\gamma)}$, where $\nu_o(\gamma) = \min\{\nu_o(\gamma_k); k = 1, \cdots, n\}$. Define the type of point w_0 by $T_\Omega(w_0) = \sup\{C_\Omega(\gamma, w_0); \text{all } \gamma \text{ with } \gamma(0) = w_0\}$ and finally, the type of Ω is $T_\Omega = \sup\{T_\Omega(w_0); w_0 \in b\Omega\}$. In the case of the L^2-norm, Catlin [2] showed that if there is a curve V parametrized by γ through $w_0 \in b\Omega$, where $\Omega \subset \mathbb{C}^n$ and (1.2) holds, then $\epsilon \leq \frac{1}{C_\Omega(\gamma, w_0)}$. In Hölder norms, McNeal [12] proved that if, with an additional assumption, Ω admits a holomorphic support function at $w_0 \in b\Omega$ and (1.3) holds, then $\epsilon \leq \frac{1}{C_\Omega(\gamma, w_0)}$.

There is the third notion of type, the "Bloom-Graham" type, $T_{BG}(w_0)$. It turns out that $T_{BG}(w_0)$ is the maximal order of contact of smooth $(n-1)$-dimensional complex submanifold. Thus, it follows that for any $w_0 \in b\Omega$, $T_{BG}(w_0) \leq T^{reg}_\Omega(w_0) \leq T_\Omega(w_0)$. Krantz [11] showed that if $T_{BG}(w_0) = m$, then $\epsilon \leq \frac{1}{m}$.

In this paper we present geometric conditions that must hold if Hölder estimate of order ϵ is valid in a neighborhood of $w_0 \in b\Omega$ in $\mathbb{C}^3$. The main result is the following theorem:

Theorem 1.1. *Let $\Omega = \{R(w) < 0\}$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$. Suppose that there is a 1-dimensional smooth analytic variety V passing through w_0 such that for all $w \in V$, w sufficiently close to w_0,*

$$|R(w)| \leq C|w - w_0|^\eta,$$

where $\eta > 0$. If there exists neighborhood W of w_0 so that for all $\alpha \in L^{0,1}_\infty(\Omega)$ with $\bar\partial\alpha = 0$, there is a $u \in \Lambda^\epsilon(W \cap \overline{\Omega})$ and $C > 0$ such that $\bar\partial u = \alpha$ and

$$\|u\|_{\Lambda^\epsilon(W\cap\overline{\Omega})} \leq C\|\alpha\|_{L_\infty(\Omega)},$$

then $\epsilon \leq \frac{1}{\eta}$.

Corollary 1.2. $\epsilon \leq \frac{1}{T^{reg}_\Omega(w_0)}$.

Remark 1.3.

i) *If $T_{BG}(w_0) = +\infty$, Krantz's result [11] holds for any $m > 0$ and we conclude $\epsilon \leq \frac{1}{m} \leq \frac{1}{\eta}$ for large m. Thus we can assume $T_{BG}(w_0) = m < \infty$. Furthermore, since $\epsilon \leq \frac{1}{m}$, we can assume $m < \eta$ in the rest of this paper.*

ii) *Theorem 1.1 improves the results by Krantz [11] and McNeal [12] in the sense that we obtain sharp result since $\eta > m$ and do not assume the existence of a holomorphic support function. Note that the existence of holomorphic support function is satisfied for restricted domains (see the Kohn-Nirenberg Domain[10]).*

To prove Theroem 1.1, the key components are the complete analysis of the local geometry near $w_0 \in b\Omega$ (Section 2) and the construction of a bounded holomorphic function with large nontangential derivative near the boundary point (Section 3). In Section 2, we construct special holomorphic coordinates about w_0 which are adapted to both Bloom-Graham type and the order of contact of V. Then, we use the truncation technique developed in [3] to deal with two dimensional slices of the domain. In Section 3, by using the holomorphic function constructed by Catlin [4] on two dimensional slice, we construct a bounded holomorphic function f with a large nontangential derivative defined locally up to the boundary in $\mathbb{C}^3$. Finally, in Section 4, we prove Theorem 1.1 by using the constructed holomorphic function.

2 Special coordinates

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ with a smooth defining function R and let $w_0 \in b\Omega$. Since $dR(w_0) \neq 0$, clearly we can assume that $\frac{\partial R}{\partial w_3}(w) \neq 0$ for all w in a small neighborhood W about w_0. Furthermore, we may assume that $w_0 = 0$. In Theorem 2.1, we construct a special coordinate near w_0 which changes the given smooth holomorphic curve into the z_1 axis and have a nonzero term along the z_2 axis when $z_1 = 0$.

Theorem 2.1. *Let $\Omega = \{w; R(w) < 0\}$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ and let $T_{BG}(0) = m$, where $0 \in b\Omega$. Suppose that there is a smooth 1-dimensional complex analytic variety V passing through 0 such that for all $w \in V, w$ sufficiently close to 0,*

$$|R(w)| \leq C|w|^\eta, \tag{2.1}$$

where $\eta > 0$. Then there is a holomorphic coordinate system (z_1, z_2, z_3) about 0 with $w = \Psi(z)$ such that

(i) $r(z) = R \circ \Psi(z) = \mathrm{Re} z_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}}^{\eta} a_{\alpha,\beta} z'^{\alpha} \bar{z}'^{\beta} + \mathcal{O}(|z_3||z| + |z'|^{\eta+1}),$

(ii) $|r(t, 0, 0)| \lesssim |t|^\eta$

(iii) $a_{0,\alpha_2,0,\beta_2} \neq 0$ *with $\alpha_2 + \beta_2 = m$ for some $\alpha_2 > 0, \beta_2 > 0$,*

where $z' = (z_1, z_2)$, *and* $z = (z_1, z_2, z_3)$.

Note that η is a positive integer since V is a smooth 1-dimensional complex analytic variety. To construct the special coordinate in Theorem 2.1, we start with a similar coordinate about 0 in $\mathbb{C}^3$ as in Proposition 1.1 in [4].

Proposition 2.2. *Let* $T_{BG}(0) = m$ *and* $\Omega = \{w \in \mathbb{C}^3; R(w) < 0\}$. *Then there is a holomorphic coordinate system* $u = (u_1, u_2, u_3)$ *with* $w = \widetilde{\Psi}(u)$ *such that the function* $\tilde{R}$, *given by* $\widetilde{R}(u) = R \circ \widetilde{\Psi}(u)$, *satisfies*

$$\widetilde{R}(u) = \mathrm{Re} u_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}}^{\eta} b_{\alpha,\beta} u'^{\alpha} \bar{u}'^{\beta} + \mathcal{O}(|u_3||u| + |u'|^{\eta+1}), \tag{2.2}$$

where $u' = (u_1, u_2)$, *and where* $b_{\alpha,\beta} \neq 0$ *for some* α, β *with* $|\alpha| + |\beta| = m$.

Proof. Bloom and Graham [1] showed that $T_{BG}(w_0) = m$ if and only if there exists coordinate with w_0 equal to the origin in $\mathbb{C}^3$ and $b_{\alpha,\beta} \neq 0$ for some α, β with $|\alpha| + |\beta| = m$ such that

$$R(w) = \mathrm{Re} w_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}} b_{\alpha,\beta} w'^{\alpha} \bar{w}'^{\beta} + \mathcal{O}(|w_3||w| + |w'|^{m+1}),$$

where $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ and $w' = (w_1, w_2)$.

Now assume that we have defined $\phi^l : \mathbb{C}^3 \to \mathbb{C}^3$ so that there exist numbers $b_{\alpha,\beta}$ for $|\alpha|, |\beta| > 0$ and $|\alpha| + |\beta| < l + 1$ with $l > m$ so that $R_l = R \circ \phi^l$ satisfies

$$R_l(v) = \mathrm{Re} v_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}}^{l} b_{\alpha,\beta} v'^{\alpha} \bar{v}'^{\beta} + \mathcal{O}(|v_3||v| + |v'|^{l+1}), \tag{2.3}$$

where $v' = (v_1, v_2)$ and $v = (v_1, v_2, v_3)$.

If we define

$$\phi^{l+1}(u) = \left(u_1, u_2, u_3 - \sum_{|\alpha|=l+1} \frac{2}{\alpha!} \frac{\partial^{l+1} R_l}{\partial v'^{\alpha}}(0) u'^{\alpha} \right),$$

then $R_{l+1} = R_l \circ \phi^{l+1} = R \circ \phi^l \circ \phi^{l+1}$ satisfies the similar form of (2.3) with l replaced by $l+1$. Therefore, if we take $\widetilde{\Psi} = \phi^l \circ \cdots \circ \phi^{\eta}$, then $\widetilde{R} = R \circ \widetilde{\Psi}$ satisfies

$$\widetilde{R}(u) = \mathrm{Re} u_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}}^{\eta} b_{\alpha,\beta} u'^{\alpha} \bar{u}'^{\beta} + \mathcal{O}(|u_3||u| + |u'|^{\eta+1}).$$

□

From now on, without loss of generality, we may assume that $\widetilde{R}$ is R by Proposition 2.2.

Lemma 2.3. *Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) : \mathbb{C} \to V$ be a local parametrization of a one-dimensional smooth complex analytic variety V. If $|R(w)| \lesssim |w|^\eta$ for $w \in V$, then we can assume $\gamma = (\gamma_1, \gamma_2, 0)$ (i.e., γ_3 vanishes to order at least η).*

Proof. We show γ_3 vanishes to order at least η. Since $\gamma(0) = 0$, we know γ_3 vanishes to some order l. If we suppose $l < \eta$, then $\gamma_3(t) = a_l t^l + \mathcal{O}(t^{l+1})$, where $a_l \neq 0$. Then

$$
\begin{aligned}
R(\gamma(t)) &= \mathrm{Re}\gamma_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0, |\beta|>0}}^{\eta} b_{\alpha,\beta} \gamma_1^{\alpha_1} \bar{\gamma}_1^{\beta_1} \gamma_2^{\alpha_2} \bar{\gamma}_2^{\beta_2} + \mathcal{O}(|\gamma_3||\gamma| + |\gamma|^{\eta+1}) \\
&= \left(\frac{a_l}{2} t^l + \frac{\bar{a}_l}{2} \bar{t}^l \right) + \left(\sum_{\substack{j+k=m \\ j>0, k>0}}^{\eta} c_{jk} t^j \bar{t}^k \right) + \mathcal{O}(|t|^{l+1}).
\end{aligned}
$$

Note that the first parenthesis consists of order l pure terms and the summation part consists of the mixed terms. The first one is essentially $|t|^l$ with $l < \eta$, so if we want to improve on the order of contact, then some terms of the summation part must cancel it. However, it is impossible because the summation part has all mixed terms. This contradicts our assumption $|r \circ \gamma(t)| \lesssim |t|^\eta$. Therefore, γ_3 vanishes to order at least η. $\square$

Let $A(u_1, u_2) = \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0, |\beta|>0}} b_{\alpha,\beta} u'^\alpha \bar{u}'^\beta$ be the homogeneous polynomial part of order m in the summation part of (2.2). In the following lemma, we show that there is some nonzero mixed term along some direction in $\mathbb{C}^2$.

Lemma 2.4. *Consider $A(hz, z)$ for all $h, z \in \mathbb{C}$. Then there is some $h \in \mathbb{C}$ such that*

$$
\frac{\partial^m A}{\partial z^j \partial \bar{z}^k}(0,0) \neq 0, \quad \textit{for} \quad j, k > 0.
$$

Proof. Suppose that for all h, $A(hz, z) = P(h)z^m + \overline{P(h)z^m}$. Since $A(hz, z)$ is a polynomial in $z, \bar{z}, h$ and $\bar{h}$ and $\frac{\partial^m A}{\partial z^m} = m!P(h)$, $P(h)$ is a polynomial. Let $P(h) = \sum a_{j,k} h^j \bar{h}^k$. Now, we have $A(hz, z) = \sum a_{j,k} h^j \bar{h}^k z^m + \sum \bar{a}_{j,k} \bar{h}^j h^k \bar{z}^m$. Since $u_1 = hz$ and $u_2 = z$, we have $h = \frac{u_1}{u_2}$ and $z = u_2$. Therefore, $A(u_1, u_2) = \sum a_{j,k} (\frac{u_1}{u_2})^j (\frac{\bar{u}_1}{\bar{u}_2})^k u_2^m + \sum \bar{a}_{j,k} (\frac{\bar{u}_1}{\bar{u}_2})^j (\frac{u_1}{u_2})^k \bar{u}_2^m$. This forces j and k to be 0 because $A(u_1, u_2)$ is a polynomial. Therefore, we have $A(hz, z) = a_{0,0} z^m + \bar{a}_{0,0} \bar{z}^m$. This means $A(u_1, u_2) = a_{0,0} {u_2}^m + \bar{a}_{0,0} \bar{u}_2^m$. However, this contradicts $b_{\alpha,\beta} \neq 0$ for some α, β with $|\alpha|, |\beta| > 0$ and $|\alpha| + |\beta| = m$ in (2.2). $\square$

Now, we prove Theorem 2.1

Proof of Theorem 2.1. We may assume ${\gamma_1}'(0) \neq 0$, and hence, after reparametrization, we can write $\gamma(t) = (t, \gamma_2(t), 0)$. Now, define

$$
u = \Psi_1(v) = (v_1, v_2 + \gamma_2(v_1), v_3).
$$

Since $\gamma_2(t) = \mathcal{O}(|t|)$ is holomorphic, (2.2) means

$$r_1(v) = R \circ \Psi_1(v) = \mathrm{Re} v_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}}^{\eta} b_{\alpha,\beta} v_1^{\alpha_1} \bar{v}_1^{\beta_1} (v_2 + \gamma_2(v_1))^{\alpha_2} \overline{(v_2 + \gamma_2(v_1))}^{\beta_2} + E_1(v)$$

$$= \mathrm{Re} v_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}}^{\eta} c_{\alpha,\beta} v_1^{\alpha_1} \bar{v}_1^{\beta_1} v_2^{\alpha_2} \bar{v}_2^{\beta_2} + E_1(v), \quad \text{where} \quad E_1(v) = \mathcal{O}(|v_3||v| + |v'|^{\eta+1}).$$

Note that $T_{BG} = m$ means $c_{\alpha,\beta} \neq 0$ for some $\alpha, \beta > 0$ with $|\alpha| + |\beta| = m$. Now, we fix h in lemma 2.4 and define

$$v = \Psi_2(z) = (z_1 + hz_2, z_2, z_3).$$

Then, we have

$$\begin{aligned} r(z) &= r_1 \circ \Psi_2(z) = R \circ \Psi_1 \circ \Psi_2(z) \\ &= \mathrm{Re} z_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}}^{\eta} c_{\alpha,\beta} (z_1 + hz_2)^{\alpha_1} \overline{(z_1 + hz_2)}^{\beta_1} z_2^{\alpha_2} \bar{z}_2^{\beta_2} + E_1(z), \qquad (2.4) \\ &= \mathrm{Re} z_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|>0,|\beta|>0}}^{\eta} a_{\alpha,\beta} z_1^{\alpha_1} \bar{z}_1^{\beta_1} z_2^{\alpha_2} \bar{z}_2^{\beta_2} + E_1(z), \qquad (2.5) \end{aligned}$$

where $a_{\alpha,\beta}$ is a polynomial of h and $\bar{h}$, and where $E_1(z) = \mathcal{O}(|z_3||z| + |z'|^{\eta+1})$. Let $\Psi = \Psi_1 \circ \Psi_2$. Then we have $r(z) = R \circ \Psi$ and (2.5) shows (i) of Theorem 2.1. Furthermore, since $|r(t, 0, 0)| = |R \circ \Psi(t, 0, 0)| = |R(\gamma(t))| \lesssim |t|^\eta$, this proves part (ii). For (iii), if we consider $r(0, z_2, 0)$ and (2.4), we have

$$r(0, z_2, 0) = A(hz_2, z_2) + \sum_{\substack{|\alpha|+|\beta|=m+1 \\ |\alpha|>0,|\beta|>0}}^{\eta} c_{\alpha,\beta} (hz_2)^{\alpha_1} \overline{(hz_2)}^{\beta_1} z_2^{\alpha_2} \bar{z}_2^{\beta_2} + \mathcal{O}(|z_2|^{\eta+1}).$$

Then Lemma 2.4 means

$$\frac{\partial^m r}{\partial {z_2}^{\alpha_2} \partial {\bar{z}_2}^{\beta_2}}(0) = \frac{\partial^m A}{\partial {z_2}^{\alpha_2} \partial {\bar{z}_2}^{\beta_2}}(0, 0) \neq 0$$

for some $\alpha_2, \beta_2 > 0$ with $\alpha_2 + \beta_2 = m$. Since $\frac{\partial^m r}{\partial {z_2}^{\alpha_2} \partial {\bar{z}_2}^{\beta_2}}(0) = \alpha_2! \beta_2! a_{0,\alpha_2,0,\beta_2}$ in (2.5), this completes the proof. $\square$

Catlin [4] constructed a bounded holomorphic funtion with a large derivative near a finite type point in the boundary of pseudoconvex domain in $\mathbb{C}^2$. To construct a similar function in $\mathbb{C}^3$, we will use the function constructed by Catlin. In order to achieve this

goal, as a first step, we need to consider two dimensional slice with respect to the z_2 and z_3 variables when z_1 is fixed at some point. For this, we consider the representative terms in the summation part of (i) of Theorem 2.1.

Let

$$\Gamma = \{(\alpha, \beta); a_{\alpha,\beta} \neq 0, m \leq |\alpha| + |\beta| \leq \eta \text{ and } |\alpha|, |\beta| > 0\}$$
$$S = \{(p, q); \alpha_1 + \beta_1 = p, \alpha_2 + \beta_2 = q \text{ for some } (\alpha, \beta) \in \Gamma\} \cup \{(\eta, 0)\}.$$

Then there is an positive integer N such that $(p_\nu, q_\nu) \in S$ for $\nu = 0, \cdots, N$ and $\eta_\nu, \lambda_\nu > 0$ for $\nu = 1, \cdots, N$ satisfying

(1) $(p_0, q_0) = (\eta, 0), (p_N, q_N) = (0, m), \lambda_N = m, \eta_1 = \eta,$

(2) $p_0 > p_1 > \cdots > p_N$ and $q_0 < q_1 < \cdots < q_N$,

(3) $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ and $\eta_1 > \eta_2 > \cdots > \eta_N$,

(4) $\frac{p_{\nu-1}}{\eta_\nu} + \frac{q_{\nu-1}}{\lambda_\nu} = 1$ and $\frac{p_\nu}{\eta_\nu} + \frac{q_\nu}{\lambda_\nu} = 1$ and

(5) $a_{\alpha,\beta} = 0$ if $\frac{\alpha_1+\beta_1}{\eta_\nu} + \frac{\alpha_2+\beta_2}{\lambda_\nu} < 1$ for each $\nu = 1, \cdots, N$.

Note that if $1 \leq l \leq m$, then $q_{\nu-1} < l \leq q_\nu$ for some $\nu = 1, \cdots, N$. Let L_ν be the line segment from $(p_{\nu-1}, q_{\nu-1})$ to (p_ν, q_ν) for each $\nu = 1, \cdots, N$ and set $L = L_1 \cup L_2 \cup \cdots \cup L_N$. Define

- $\Gamma_L = \{(\alpha, \beta) \in \Gamma; \alpha + \beta \in L\}$.
- $t_l = \begin{cases} \eta & \text{if } l = 0 \\ \eta_\nu \left(1 - \frac{l}{\lambda_\nu}\right) & \text{if } q_{\nu-1} < l \leq q_\nu \text{ for some } \nu. \end{cases}$

Note that $(p_{\nu-1}, q_{\nu-1}), (t_l, l)$ and (p_ν, q_ν) are collinear points in the first quadrant of the plane and η_ν and λ_ν are the x, y-intercepts of the line.

Now, we want to show that for each element (p_ν, q_ν) with $\nu = 1, \cdots, N$, there is some (α, β) allowing a mixed term in the z_2 variable. To show this, we need to use a variant of the notations and the results from Lemma 4.1 and Proposition 4.4 in [3]. For t with $0 < t < 1$ and each $\nu = 1, \cdots, N$, define a family of a truncation map $H_t^\nu : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$H_t^\nu(z_1, z_2, z_3) = (t^{(1/\eta_\nu)} z_1, t^{(1/\lambda_\nu)} z_2, t z_3).$$

Set $r_t^\nu = t^{-1}(H_t^{\nu *} r)$ and $\widetilde{r}^\nu = \lim_{t \to 0} r_t^\nu$. Note that

$$\widetilde{r}^\nu(z) = \mathrm{Re} z_3 + \sum_{\substack{\frac{\alpha_1+\beta_1}{\eta_\nu} + \frac{\alpha_2+\beta_2}{\lambda_\nu} = 1 \\ (\alpha,\beta) \in \Gamma_L}} a_{\alpha_1,\alpha_2,\beta_1,\beta_2} z_1^{\alpha_1} \bar{z}_1^{\beta_1} z_2^{\alpha_2} \bar{z}_2^{\beta_2}.$$

Let r and $\widetilde{r}^\nu$ be a defining function of Ω and $\widetilde{\Omega}_\nu$ near 0. Observe that if Ω is pseudoconvex, then $\widetilde{\Omega}_\nu$ must also be pseudoconvex, for $\widetilde{r}^\nu$ equals the limit in the C^∞-topology of r_t^ν, which for each t is the defining function of a pseudoconvex domain. For a fixed (z_1, z_2), choose z_3 so that $\widetilde{r}^\nu(z_1, z_2, z_3) = 0$. Let that point z. Since the Hessian of $\widetilde{r}^\nu$ is nonnegative in the tangential directions at z, it follows that the Hessian of $\widetilde{r}^\nu$ is nonnegative at z. This means $\widetilde{r}^\nu$ is plurisubharmonic.

Lemma 2.5. *Consider r in (i) of Theorem 2.1. Then for each $\nu = 1, \cdots, N$, there is $(\alpha^\nu, \beta^\nu) \in \Gamma_L$ with $\alpha_2^\nu > 0, \beta_2^\nu > 0$ and $\alpha^\nu + \beta^\nu = (p_\nu, q_\nu)$.*

Proof. Consider $\widetilde{r}^\nu$, which is plurisubharmonic. Now, consider

$$\widetilde{(\widetilde{r}^\nu)}^{\nu+1} = \lim_{t\to 0} t^{-1}(H_t^{\nu+1*}\widetilde{r}^\nu).$$

This is also plurisubharmonic. Since (p_ν, q_ν) is the unique point with $L_\nu \cap L_{\nu+1}$ (i.e., $\frac{p_\nu}{\eta_\nu} + \frac{q_\nu}{\lambda_\nu} = 1$ and $\frac{p_\nu}{\eta_{\nu+1}} + \frac{q_\nu}{\lambda_{\nu+1}} = 1$), we have

$$\widetilde{(\widetilde{r}^\nu)}^{\nu+1} = \mathrm{Re} z_3 + \sum_{\substack{\alpha+\beta=(p_\nu,q_\nu)\\(\alpha,\beta)\in\Gamma_L}} a_{\alpha_1,\alpha_2,\beta_1,\beta_2} z_1^{\alpha_1}\bar{z}_1^{\beta_1} z_2^{\alpha_2}\bar{z}_2^{\beta_2}. \tag{2.6}$$

In particular, $(\alpha, \beta) \in \Gamma_L$ means $|\alpha|, |\beta| > 0$. Suppose that $\widetilde{(\widetilde{r}^\nu)}^{\nu+1}$ has no terms with both $\alpha_2 > 0$ and $\beta_2 > 0$ in (2.6) (i.e., no mixed terms in z_2 variable). Thus

$$\widetilde{(\widetilde{r}^\nu)}^{\nu+1} = \mathrm{Re} z_3 + P_{q_\nu}(z_1){z_2}^{q_\nu} + \overline{P_{q_\nu}(z_1){z_2}^{q_\nu}}$$

where $P_{q_\nu}(z_1) = \sum_{\alpha_1+\beta_1=p_\nu} c_{\alpha_i,\beta_i}{z_1}^{\alpha_1}\bar{z}_1^{\beta_1}$ with $\beta_1 > 0$. By the plurisubharmonicity of $\widetilde{(\widetilde{r}^\nu)}^{\nu+1}$,

$$\widetilde{(\widetilde{r}^\nu)}^{\nu+1}_{11}\widetilde{(\widetilde{r}^\nu)}^{\nu+1}_{22} - \widetilde{(\widetilde{r}^\nu)}^{\nu+1}_{12}\widetilde{(\widetilde{r}^\nu)}^{\nu+1}_{21} = -|q_\nu \frac{\partial P_{q_\nu}}{\partial \bar{z}_1}(z_1){z_2}^{q_\nu-1}|^2 \geq 0,$$

where $\widetilde{(\widetilde{r}^\nu)}^{\nu+1}_{ij} = \frac{\partial^2 \widetilde{(\widetilde{r}^\nu)}^{\nu+1}}{\partial z_i \partial \bar{z}_j}$ for $i, j = 1, 2$. Therefore, we have $\frac{\partial P_{q_\nu}}{\partial \bar{z}_1}(z_1) = 0$. This means $P_{q_\nu}(z_1)$ is holomorphic. This contradicts the fact that $P_{q_\nu}(z_1) = \sum_{\alpha_1+\beta_1=p_\nu} c_{\alpha_i,\beta_i}{z_1}^{\alpha_1}\bar{z}_1^{\beta_1}$ with $\beta_1 > 0$. ∎

Now, we define these special terms with respect to the z_2 variable. Let

$$\Lambda = \{(\alpha, \beta) \in \Gamma_L; \alpha + \beta = (p_\nu, q_\nu), \alpha_2 > 0, \beta_2 > 0, \nu = 1, \cdots, N\}.$$

Then we represent the expression of r in terms of these terms.

Proposition 2.6. *The defining function r can be expressed as*

$$r(z) = \mathrm{Re} z_3 + \sum_{\Gamma_L - \Lambda} a_{\alpha,\beta} z'^{\alpha} \bar{z}'^{\beta} + \sum_{\nu=1}^{N} \sum_{\substack{\alpha_2+\beta_2=q_\nu \\ \alpha_2>0,\beta_2>0}} M_{\alpha_2,\beta_2}(z_1) z_2^{\alpha_2} \bar{z}_2^{\beta_2} + E_2(z), \tag{2.7}$$

where $M_{\alpha_2,\beta_2}(z_1) = \sum\limits_{\alpha_1+\beta_1=p_\nu} a_{\alpha,\beta} z_1^{\alpha_1} \bar{z}_1^{\beta_1}$ *and* $E_2(z) = \mathcal{O}(|z_3||z| + \sum_{\nu=1}^{N} \sum_{l=q_{\nu-1}}^{q_\nu} |z_1|^{[t_l]+1}|z_2|^l + |z_2|^{m+1})$.

Proof. By theorem 2.1, we have

$$r(z) = \mathrm{Re} z_3 + \sum_{\Gamma_L} a_{\alpha,\beta} z'^{\alpha} \bar{z}'^{\beta} + \sum_{\Gamma-\Gamma_L} a_{\alpha,\beta} z'^{\alpha} \bar{z}'^{\beta} + \mathcal{O}(|z_3||z| + |z'|^{\eta+1}). \tag{2.8}$$

Suppose that $(k,l) = (\alpha_1+\beta_1, \alpha_2+\beta_2)$ for some $(\alpha,\beta) \in \Gamma - \Gamma_L$. Then, we consider two cases; $1 \le l \le m$ and $m < l < \eta$. If $1 \le l \le m$, there is a unique $\nu = 1, \cdots, N$ so that $q_{\nu-1} < l \le q_\nu$ and $t_l = \eta_\nu \left(1 - \frac{l}{\lambda_\nu}\right)$. Since $(k,l) = (\alpha_1+\beta_1, \alpha_2+\beta_2)$ for some $(\alpha,\beta) \in \Gamma - \Gamma_L$, $\frac{k}{\eta_\nu} + \frac{l}{\lambda_\nu} > 1$. This gives $t_l = \eta_\nu \left(1 - \frac{l}{\lambda_\nu}\right) < k$. Since k is an integer, $[t_l] + 1 \le k$. Thus, we have $|z_1|^k|z_2|^l \le |z_1|^{[t_l]+1}|z_2|^l$ for each $l = 1, \cdots, m$. On the other hand, if $(k,l) = (\alpha_1+\beta_1, \alpha_2+\beta_2)$ for some $(\alpha,\beta) \in \Gamma - \Gamma_L$ and $m < l < \eta$, then $|z_1|^k|z_2|^l \le |z_1|^k|z_2|^{m+1} \le |z_2|^{m+1}$ for small z_1 and z_2. Since $|z'|^{\eta+1} \approx |z_1|^{\eta+1} + |z_2|^{\eta+1}$, it follows that $\sum_{\Gamma-\Gamma_L} a_{\alpha,\beta} z'^{\alpha} \bar{z}'^{\beta} + \mathcal{O}(|z_3||z| + |z'|^{\eta+1}) = \mathcal{O}(|z_3||z| + \sum_{\nu=1}^{N} \sum_{l=q_{\nu-1}}^{q_\nu} |z_1|^{[t_l]+1}|z_2|^l + |z_2|^{m+1})$. Therefore, $r(z)$ in (2.8) is represented as

$$\mathrm{Re} z_3 + \sum_{\Gamma_L} a_{\alpha,\beta} z'^{\alpha} \bar{z}'^{\beta} + \mathcal{O}(|z_3||z| + \sum_{\nu=1}^{N} \sum_{l=q_{\nu-1}}^{q_\nu} |z_1|^{[t_l]+1}|z_2|^l + |z_2|^{m+1}). \tag{2.9}$$

Now, apply $\Gamma_L = (\Gamma_L - \Lambda) \cup \Lambda$ for the second part of summation in (2.8). $\square$

Remark 2.7.

i) *$M_{\alpha_2,\beta_2}(z_1)$ is not identically zero for $\alpha_2+\beta_2 = q_\nu$ and the homogeneous polynomial is of order p_ν for each $\nu = 1, \cdots, N-1$.*

ii) *If $\nu = N$, then $|M_{\alpha_2,\beta_2}(z_1)|$ is a nonzero constant for all $\alpha_2, \beta_2 > 0$ with $\alpha_2+\beta_2 = m = q_N$ since $p_N = 0$.*

iii) *Since $M_{\alpha_2,\beta_2}(z_1)$ is a homogeneous polynomial of order $p_\nu, \nu = 1, \cdots, N$, in z_1-variable, there are $\theta_0 \in [0, 2\pi]$ and a small constant $c > 0$ such that $|M_{\alpha_2,\beta_2}(\tau e^{i\theta})| \ne 0$ for all $|\theta - \theta_0| < c$ and $0 < \tau \le 1$. In particular, if we take $d = e^{i\theta_0}$ and $\tau = \delta^{\frac{1}{\eta}}$ we have $|M_{\alpha_2,\beta_2}(d\delta^{\frac{1}{\eta}})| \approx \delta^{\frac{p_\nu}{\eta}}$ for all $\alpha_2+\beta_2 = q_\nu$ with all $\nu = 1, \cdots, N$.*

3 The construction of bounded holomorphic function with large derivative near the boundary

Let $z_1 = d\delta^{\frac{1}{\eta}}$. Then, we get a complex two dimensional slice. After the holomorphic coordinate change as Proposition 1.1 in [4], we can define a bounded holomorphic function with a large nontangential derivative as in [4] on the slice. In this section, first, we construct a holomorphic coordinate system in $\mathbb{C}^3$ to exactly fit the holomorphic coordinate system as in proposition 1.1 of [4] when z_1 is fixed as $d\delta^{\frac{1}{\eta}}$. Second, we show that the holomorphic function defined on the slice is also well-defined on a family of slices along the small neighborhood of $z_1 = d\delta^{\frac{1}{\eta}}$. To show the well-definedness of the holomorphic function up to boundary in $\mathbb{C}^3$, we need the estimates of derivatives. Let's denote $U\big|_{z_1=d\delta^{\frac{1}{\eta}}} = U \cap \{(d\delta^{\frac{1}{\eta}}, z_2, z_3)\}$ and let $\widetilde{e}_\delta = (d\delta^{\frac{1}{\eta}}, 0, e_\delta)$ satisfy $r(\widetilde{e}_\delta) = 0$. Since $\frac{\partial r}{\partial z_3}(0) \neq 0$, clearly $\frac{\partial r}{\partial z_3}(\widetilde{e}_\delta) \neq 0$. We start with the similar argument as Proposition 1.1 in [4].

Proposition 3.1. *For $\widetilde{e}_\delta \in U\big|_{z_1=d\delta^{\frac{1}{\eta}}}$, there exists a holomorphic coordinate system $(z_2, z_3) = \Phi_{\widetilde{e}_\delta}(\zeta'') = (\zeta_2, \Phi_3(\zeta''))$ such that in the new coordinate $\zeta'' = (\zeta_2, \zeta_3)$ defined by*

$$\Phi_{\widetilde{e}_\delta}(\zeta'') = \left(\zeta_2,\ e_\delta + \left(\frac{\partial r}{\partial z_3}(\widetilde{e}_\delta)\right)^{-1}\left(\frac{\zeta_3}{2} - \sum_{l=2}^{m} c_l(\widetilde{e}_\delta)\zeta_2^l - \frac{\partial r}{\partial z_2}(\widetilde{e}_\delta)\zeta_2\right)\right), \tag{3.1}$$

the function $\rho(d\delta^{\frac{1}{\eta}}, \zeta'') = r(d\delta^{\frac{1}{\eta}}, z'') \circ \Phi_{\widetilde{e}_\delta}(\zeta'')$ satisfies

$$\rho(d\delta^{\frac{1}{\eta}}, \zeta'') = \mathrm{Re}\zeta_3 + \sum_{\substack{j+k=2\\ j,k>0}}^{m} a_{j,k}(\widetilde{e}_\delta)\zeta_2^j\bar{\zeta}_2^k + \mathcal{O}(|\zeta_3||\zeta''| + |\zeta_2|^{m+1}), \tag{3.2}$$

where $z'' = (z_2, z_3)$.

Proof. For $\widetilde{e}_\delta \in U\big|_{z_1=d\delta^{\frac{1}{\eta}}}$, define

$$\Phi^1_{\widetilde{e}_\delta}(w'') = \left(w_2,\ e_\delta + \left(\frac{\partial r}{\partial z_3}(\widetilde{e}_\delta)\right)^{-1}\left(\frac{w_3}{2} - \frac{\partial r}{\partial z_2}(\widetilde{e}_\delta)w_2\right)\right). \tag{3.3}$$

Then we have

$$\rho_2(d\delta^{\frac{1}{\eta}}, w'') = r(d\delta^{\frac{1}{\eta}}, z'') \circ \Phi^1_{\widetilde{e}_\delta}(w'') = \mathrm{Re}w_3 + \mathcal{O}(|w''|^2), \tag{3.4}$$

where $w'' = (w_2, w_3)$. Now assume that we have defined $\Phi^{l-1}_{\widetilde{e}_\delta} : \mathbb{C}^2 \to \mathbb{C}^2$ so that there exist numbers $a_{j,k}$ for $j,k > 0$ and $j+k < l$ so that $\rho_l(d\delta^{\frac{1}{\eta}}, w'') = r(d\delta^{\frac{1}{\eta}}, z'') \circ \Phi^{l-1}_{\widetilde{e}_\delta}(w'')$ satisfies

$$\rho_l(d\delta^{\frac{1}{\eta}}, w'') = \mathrm{Re}w_3 + \sum_{\substack{j+k=2\\ j,k>0}}^{l-1} a_{j,k}(\widetilde{e}_\delta)w_2^j\bar{w}_2^k + \mathcal{O}(|w_3||w''| + |w_2|^l),$$

where $w'' = (w_2, w_3)$. If we define $\Phi^l_{\widetilde{e}_\delta} = \Phi^{l-1}_{\widetilde{e}_\delta} \circ \phi^l$, where

$$\phi^l(\zeta'') = \left(\zeta_2,\ \zeta_3 - \frac{2}{l!}\frac{\partial^l \rho_l}{\partial w_2^l}(d\delta^{\frac{1}{\eta}}, 0, 0)\zeta_2^l\right). \tag{3.5}$$

then

$$\rho_{l+1}(d\delta^{\frac{1}{\eta}}, \zeta'') = \rho_l \circ \phi^l(\zeta'') = r(d\delta^{\frac{1}{\eta}}, z'') \circ \Phi^l_{\widetilde{e}_\delta}(\zeta'') \tag{3.6}$$

satisfies

$$\rho_{l+1}(d\delta^{\frac{1}{\eta}}, \zeta'') = \mathrm{Re}\zeta_3 + \sum_{\substack{j+k=2\\ j,k>0}}^{l} a_{j,k}(\widetilde{e}_\delta)\zeta_2^j \bar{\zeta}_2^k + \mathcal{O}(|\zeta_3||\zeta''| + |\zeta_2|^{l+1}),$$

where $\zeta'' = (\zeta_2, \zeta_3)$. Therefore, if we choose $\Phi_{\widetilde{e}_\delta} = \Phi^m_{\widetilde{e}_\delta} = \Phi^{m-1}_{\widetilde{e}_\delta} \circ \phi^m = \cdots = \Phi^1_{\widetilde{e}_\delta} \circ \phi^2 \circ \cdots \circ \phi^m$, then $\rho = \rho_{m+1} = \rho_m \circ \phi^m = r \circ \Phi_{\widetilde{e}_\delta}$. This shows (3.1) and (3.2), where $c_l(\widetilde{e}_\delta)$ is defined by

$$c_l(\widetilde{e}_\delta) = \frac{1}{l!}\frac{\partial^l \rho_l}{\partial w_2^l}(d\delta^{\frac{1}{\eta}}, 0, 0). \tag{3.7}$$

$\square$

As in [4], we set

$$A_l(\widetilde{e}_\delta) = \max\{|a_{j,k}(\widetilde{e}_\delta)|; j + k = l\}, \qquad l = 2, \cdots, m \tag{3.8}$$

and

$$\tau(\widetilde{e}_\delta, \delta) = \min\left\{\left(\frac{\delta}{A_l(\widetilde{e}_\delta)}\right)^{1/l}; 2 \le l \le m\right\} \tag{3.9}$$

As we will see later (Remark 3.4), we have $A_m(\widetilde{e}_\delta) \neq 0$ since $|A_m(\widetilde{e}_\delta)| \ge c_m > 0$, where $\delta > 0$ is sufficiently small. This means

$$\tau(\widetilde{e}_\delta, \delta) \lesssim \delta^{\frac{1}{m}}.$$

Define

$$R_\delta(\widetilde{e}_\delta) = \{\zeta'' \in \mathbb{C}^2; |\zeta_2| < \tau(\widetilde{e}_\delta, \delta), |\zeta_3| < \delta\}. \tag{3.10}$$

Before estimating the derivative of r, we estimate the size of e_δ. Since $r(\widetilde{e}_\delta) = 0$, Taylor's theorem in z_3 about e_δ gives

$$r(d\delta^{\frac{1}{\eta}}, 0, z_3) = 2\mathrm{Re}\left(\frac{\partial r}{\partial z_3}(d\delta^{\frac{1}{\eta}}, 0, e_\delta)(z_3 - e_\delta)\right) + \mathcal{O}(|z_3 - e_\delta|^2).$$

If we take $z_3 = 0$, then $|r(d\delta^{\frac{1}{\eta}}, 0, 0)| = \left|2\mathrm{Re}\left(\frac{\partial r}{\partial z_3}(d\delta^{\frac{1}{\eta}}, 0, 0)(-e_\delta)\right) + \mathcal{O}(|e_\delta|^2)\right| \approx |e_\delta|$ since $|e_\delta| \ll 1$ and $|\frac{\partial r}{\partial z_3}| \approx 1$ near 0. Therefore ii) of Theorem 2.1 means $|e_\delta| \lesssim \delta$.

Lemma 3.2. *Let $l = 1, 2, \cdots, m$ and let α_2^ν and β_2^ν be positive numbers as given in Lemma 2.5 for $\nu = 1, \cdots, N$. Then the function r satisfies*

(i) $\left|\frac{\partial^l r}{\partial z_2^{\alpha_2} \partial \bar{z}_2^{\beta_2}}(\widetilde{e}_\delta)\right| \lesssim \delta^{\frac{t_l}{\eta}}$, *where* $\alpha_2, \beta_2 \geq 0$.

(ii) $\left|\frac{\partial^{q_\nu} r}{\partial z_2^{\alpha_2^\nu} \partial \bar{z}_2^{\beta_2^\nu}}(\widetilde{e}_\delta)\right| \approx \delta^{\frac{p_\nu}{\eta}}$, *where* $\alpha_2^\nu > 0$ *and* $\beta_2^\nu > 0$.

Proof. By (2.9) and $t_l < [t_l] + 1$, we have

$$\left|\frac{\partial^l r}{\partial z_2^{\alpha_2} \partial \bar{z}_2^{\beta_2}}(\widetilde{e}_\delta)\right| \lesssim \delta^{\frac{t_l}{\eta}} + |e_\delta| + \delta^{\frac{[t_l]+1}{\eta}} \lesssim \delta^{\frac{t_l}{\eta}}. \tag{3.11}$$

For (ii), note that if $l = q_\nu$, then $t_l = p_\nu$. Therefore, (2.7) gives

$$|M_{\alpha_2^\nu, \beta_2^\nu}(d\delta^{\frac{1}{\eta}})| - C_1(|e_\delta| + \delta^{\frac{p_\nu+1}{\eta}}) \leq \left|\frac{1}{\alpha_2^\nu! \beta_2^\nu!} \frac{\partial^{q_\nu} r}{\partial z_2^{\alpha_2^\nu} \partial \bar{z}_2^{\beta_2^\nu}}(\widetilde{e}_\delta)\right| \leq |M_{\alpha_2^\nu, \beta_2^\nu}(d\delta^{\frac{1}{\eta}})| + C_1(|e_\delta| + \delta^{\frac{p_\nu+1}{\eta}})$$

for some constant C_1. Since Remark 2.7 means $|M_{\alpha_2^\nu, \beta_2^\nu}(d\delta^{\frac{1}{\eta}})| \approx \delta^{\frac{p_\nu}{\eta}}$, we have

$$\left|\frac{\partial^{q_\nu} r}{\partial z_2^{\alpha_2^\nu} \partial \bar{z}_2^{\beta_2^\nu}}(\widetilde{e}_\delta)\right| \approx \delta^{\frac{p_\nu}{\eta}}.$$

□

Lemma 3.3. *Let ρ_l, ϕ^l and Φ^l be given as in (3.3)-(3.6) for $l = 2, \cdots, m+1$ and α_2^ν and β_2^ν be positive numbers as given in Lemma 2.5 for $\nu = 1, \cdots, N$. Then*

(i) $\left|\frac{\partial^k \rho_l}{\partial \zeta_2^{\alpha_2} \partial \bar{\zeta}_2^{\beta_2}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \lesssim \delta^{\frac{t_k}{\eta}}$ *for each* $k = 1, \cdots, m$.

(ii) $\left|\frac{\partial^{q_\nu} \rho_l}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \approx \delta^{\frac{p_\nu}{\eta}}$ *for each* $\nu = 1, \cdots, N$.

In particular, $|c_l(\widetilde{e}_\delta)| \lesssim \delta^{\frac{t_l}{\eta}}$, *where* $c_l(\widetilde{e}_\delta)$ *is given in (3.7).*

Proof. By induction, we prove both (i) and (ii). For part (i), let $l = 2$. Since $\rho_2(d\delta^{\frac{1}{\eta}}, \zeta'') = r(d\delta^{\frac{1}{\eta}}, z'') \circ \Phi^1_{\widetilde{e}_\delta}(\zeta'')$, by chain rule and Lemma 3.2, we have

$$\left|\frac{\partial^k \rho_2}{\partial \zeta_2^{\alpha_2} \partial \bar{\zeta}_2^{\beta_2}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \lesssim \left|\frac{\partial^k r}{\partial z_2^{\alpha_2} \partial \bar{z}_2^{\beta_2}}(\widetilde{e}_\delta)\right| + \left|\frac{\partial r}{\partial z_2}(\widetilde{e}_\delta)\right| \lesssim \delta^{\frac{t_k}{\eta}} + \delta^{\frac{t_1}{\eta}} \lesssim \delta^{\frac{t_k}{\eta}}.$$

for all $k = 1, \cdots, m$. This proves for the case $l = 2$. Now, by induction, we assume

$$\left|\frac{\partial^k \rho_l}{\partial \zeta_2^{\alpha_2} \partial \bar{\zeta}_2^{\beta_2}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \lesssim \delta^{\frac{t_k}{\eta}}$$

for all $k = 1, \cdots, m$ and $l = 2, \cdots, j$. Note that

$$\rho_{j+1}(d\delta^{\frac{1}{\eta}}, \zeta_2, \zeta_3) = \rho_j(d\delta^{\frac{1}{\eta}},\ \zeta_2,\ \zeta_3 - 2c_j(\widetilde{e}_\delta)\zeta_2^j). \tag{3.12}$$

If $k < j$, the inductive assumption gives

$$\left|\frac{\partial^k \rho_{j+1}}{\partial \zeta_2^{\alpha_2} \partial \bar{\zeta}_2^{\beta_2}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| = \left|\frac{\partial^k \rho_j}{\partial w_2^{\alpha_2} \partial \bar{w}_2^{\beta_2}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \lesssim \delta^{\frac{t_k}{\eta}}.$$

Now, let $k = j$. If $\alpha_2 > 0$ and $\beta_2 > 0$, we have the same result as the previous one. Otherwise, $\frac{\partial^j \rho_{j+1}}{\partial \zeta_2^j}(d\delta^{\frac{1}{\eta}}, 0, 0) = \frac{\partial^j \rho_j}{\partial w_2^j}(d\delta^{\frac{1}{\eta}}, 0, 0) - 2j!c_j(\widetilde{e}_\delta)\frac{\partial \rho_j}{\partial w_3}(d\delta^{\frac{1}{\eta}}, 0, 0) = 0$. If $k > j$, the inductive assumption gives

$$\left|\frac{\partial^k \rho_{j+1}}{\partial \zeta_2^{\alpha_2} \partial \bar{\zeta}_2^{\beta_2}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \lesssim \left|\frac{\partial^k \rho_j}{\partial w_2^{\alpha_2} \partial \bar{w}_2^{\beta_2}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| + |c_j(\widetilde{e}_\delta)| \lesssim \delta^{\frac{t_k}{\eta}} + \delta^{\frac{t_j}{\eta}} \lesssim \delta^{\frac{t_k}{\eta}}.$$

For part (ii), let $l = 2$ and apply the chain rule again to ρ_2, we have

$$\left|\frac{\partial^{q_\nu} r}{\partial z_2^{\alpha_2^\nu} \partial \bar{z}_2^{\beta_2^\nu}}(\widetilde{e}_\delta)\right| - C\left|\frac{\partial r}{\partial z_2}(\widetilde{e}_\delta)\right| \leq \left|\frac{\partial^{q_\nu} \rho_2}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \leq \left|\frac{\partial^{q_\nu} r}{\partial z_2^{\alpha_2^\nu} \partial \bar{z}_2^{\beta_2^\nu}}(\widetilde{e}_\delta)\right| + C\left|\frac{\partial r}{\partial z_2}(\widetilde{e}_\delta)\right|$$

for some constant C. Then, Lemma 3.2 means

$$\delta^{\frac{p_\nu}{\eta}} - \delta^{\frac{t_1}{\eta}} \lesssim \left|\frac{\partial^{q_\nu} \rho_2}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \lesssim \delta^{\frac{p_\nu}{\eta}} + \delta^{\frac{t_1}{\eta}}. \tag{3.13}$$

Since $1 < q_\nu$ for each $\nu = 1, \cdots, N$, it gives $p_\nu = t_{q_\nu} < t_1$. Therefore, we have

$$\left|\frac{\partial^{q_\nu} \rho_2}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \approx \delta^{\frac{p_\nu}{\eta}}.$$

This proves the statement for the case $l = 2$. By induction, assume $\left|\frac{\partial^{q_\nu} \rho_l}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \approx \delta^{\frac{p_\nu}{\eta}}$. First, consider the case when $q_\nu \leq l$. Since $\alpha_2^\nu > 0$ and $\beta_2^\nu > 0$, by the similar argument as in the proof of (i) and the by inductive assumption, we have

$$\left|\frac{\partial^{q_\nu} \rho_{l+1}}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| = \left|\frac{\partial^{q_\nu} \rho_l}{\partial w_2^{\alpha_2^\nu} \partial \bar{w}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \approx \delta^{\frac{t_{q_\nu}}{\eta}} = \delta^{\frac{p_\nu}{\eta}}.$$

Now, consider the case when $q_\nu > l$, If we take the derivative of ρ_{l+1} in (3.12) about ζ_2, the derivative related to the third component involves $c_l(\widetilde{e}_\delta)$. Therefore, we have

$$\begin{aligned}\left|\frac{\partial^{q_\nu} \rho_l}{\partial w_2^{\alpha_2^\nu} \partial \bar{w}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| - C'|c_l(\widetilde{e}_\delta)| \leq \left|\frac{\partial^{q_\nu} \rho_{l+1}}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| &\leq \left|\frac{\partial^{q_\nu} \rho_l}{\partial w_2^{\alpha_2^\nu} \partial \bar{w}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \\ &\quad + C'|c_l(\widetilde{e}_\delta)|\end{aligned}$$

for some constant C'. Therefore, the inductive assumption and part (i) means

$$\delta^{\frac{p_\nu}{\eta}} - \delta^{\frac{t_l}{\eta}} \lesssim \left| \frac{\partial^{q_\nu} \rho_{l+1}}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}} (d\delta^{\frac{1}{\eta}}, 0, 0) \right| \lesssim \delta^{\frac{p_\nu}{\eta}} - \delta^{\frac{t_l}{\eta}} \tag{3.14}$$

Since $q_\nu > l$, it means $p_\nu = t_{q_\nu} < t_l$. Thus, we have $\left| \frac{\partial^{q_\nu} \rho_{l+1}}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}} (d\delta^{\frac{1}{\eta}}, 0, 0) \right| \approx \delta^{\frac{p_\nu}{\eta}}$.

□

Finally, we show that the derivatives of ρ can be bounded from below.

Remark 3.4. *Take $\nu = N$. Since $\left| \frac{\partial^{q_\nu} \rho}{\partial \zeta_2^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}} (d\delta^{\frac{1}{\eta}}, 0, 0) \right| \approx |A_m(\widetilde{e}_\delta)|$, Lemma 3.3 means $|A_m(\widetilde{e}_\delta)| \approx 1$.*

Now, we recall some facts in [4] before showing the holomorphic function defined in the complex two dimensional slice(i.e z_1 is fixed) is well-defined when we move z_1 in a small neighborhood of $z_1 = d\delta^{\frac{1}{\eta}}$.

Theorem 3.5 (Catlin). *Suppose the defining function ρ for a pseudoconvex domain in $b\Omega \subset \mathbb{C}^2$ has the following form:*

$$\rho(\zeta) = \mathrm{Re}\zeta_2 + \sum_{\substack{j+k=2 \\ j,k>0}}^{m} a_{j,k} {\zeta_1}^j {\bar{\zeta}_1}^k + \mathcal{O}(|\zeta_2||\zeta| + |\zeta_1|^{m+1}).$$

Set

$$A_l = \max\{|a_{j,k}|; j+k=l\}, \qquad l = 2, \cdots, m.$$

and

$$J_\delta(\zeta) = (\delta^2 + |\zeta_2|^2 + \sum_{k=2}^{m} (A_k)^2 |\zeta_1|^{2k})^{\frac{1}{2}}.$$

Define

$$\Omega_{a,\delta}^{\epsilon_0} = \{\zeta; |\zeta_1| < a, |\zeta_2| < a, \rho(\zeta) < \epsilon J_\delta(\zeta)\} \quad \textit{for any small constant } a, \epsilon_0 > 0.$$

If we have $|A_m| \geq c_m > 0$ for some positive constant c_m, then there exist small constants $a, \epsilon_0 > 0$ so that for any sufficiently small $\delta > 0$, there is a L^2 holomorphic function $f \in A(\Omega_{a,\delta}^{\epsilon_0})$ satisfying $\left| \frac{\partial f}{\partial \zeta_2}(0, -\frac{b\delta}{2}) \right| \geq \frac{1}{2\delta}$ for some small constant b. Moreover, the values a and ϵ_0 depend only on the constant c_m and $C_{m+1} = \|\rho\|_{C^{m+1}(U)}$, where U is a small neighborhood of 0.

The result stated in [4] applies to a more restricted situation, but a careful examination of the proof actually implies the above result. To apply theorem 3.5 to the complex two dimensional slice, we consider the pushed out domain about $\widetilde{e}_\delta$. Let $\Phi_{\widetilde{e}_\delta}$ be the map

associated with $\widetilde{e}_\delta$ as in (3.1). Set $U''\big|_{z_1=d\delta^{\frac{1}{\eta}}} = \{\zeta'' = (\zeta_2,\zeta_3); \Phi_{\widetilde{e}_\delta}(\zeta'') \in U\big|_{z_1=d\delta^{\frac{1}{\eta}}}\}$. For all small δ, define

$$J_\delta(\zeta'') = \left(\delta^2 + |\zeta_3|^2 + \sum_{k=2}^{m} (A_k(\widetilde{e}_\delta))^2 |\zeta_2|^{2k}\right)^{\frac{1}{2}} \tag{3.15}$$

and the pushed-out domain with respect to the slice

$$\Omega^{\epsilon_0}_{a,\delta} = \{(\zeta_2,\zeta_3); |\zeta_2| < a, |\zeta_3| < a \text{ and } \rho(d\delta^{\frac{1}{\eta}}, \zeta'') < \epsilon_0 J_\delta(\zeta'')\}. \tag{3.16}$$

By Theorem 3.5, we have a L^2 holomorphic function f in $\Omega^{\epsilon_0}_{a,\delta}$ satisfying

$$\left|\frac{\partial f}{\partial \zeta_3}(0, -\frac{b\delta}{2})\right| \geq \frac{1}{2\delta}. \tag{3.17}$$

In order to show the well-definedness of the holomorphic function f when z_1 moves in a small neighborhood of $z_1 = d\delta^{\frac{1}{\eta}}$, we use $\Phi_{\widetilde{e}_\delta}$ given as in (3.1) and define

$$\Phi(\zeta_1,\zeta_2,\zeta_3) = (\zeta_1,\zeta_2,\Phi_3(\zeta)),$$

where $\Phi_3(\zeta)$ is defined by

$$\Phi_3(\zeta) = e_\delta + \left(\frac{\partial r}{\partial z_3}(\widetilde{e}_\delta)\right)^{-1} \left(\frac{\zeta_3}{2} - \sum_{l=2}^{m} c_l(\widetilde{e}_\delta)\zeta_2^l - \frac{\partial r}{\partial z_2}(\widetilde{e}_\delta)\zeta_2\right) \tag{3.18}$$

and define

$$\rho(\zeta_1,\zeta_2,\zeta_3) = r(z_1,z_2,z_3) \circ \Phi(\zeta_1,\zeta_2,\zeta_3). \tag{3.19}$$

In particular, when we fix $z_1 = d\delta^{\frac{1}{\eta}}$, we have the holomophic function f defined in the slice $\Omega^{\epsilon_0}_{a,\delta}$ satisfying (3.17). Now, we consider the domain given by the family of the pushed out domains of the slice along with ζ_1 axis and the domain in the new coordinate of Ω by Φ. Define

$$\Omega^{\epsilon_0}_{a,\delta,\zeta_1} = \{\zeta \in \mathbb{C}^3; |\zeta_1 - d\delta^{\frac{1}{\eta}}| < c\delta^{\frac{1}{\eta}}, |\zeta_2| < a, |\zeta_3| < a \text{ and } \rho(d\delta^{\frac{1}{\eta}}, \zeta'') < \epsilon_0 J_\delta(\zeta'')\}$$

and

$$\Omega_{a,\delta,\zeta_1} = \{\zeta \in \mathbb{C}^3; |\zeta_1 - d\delta^{\frac{1}{\eta}}| < c\delta^{\frac{1}{\eta}}, |\zeta_2| < a, |\zeta_3| < a \text{ and } \rho(\zeta_1, \zeta'') < 0\}$$

for some small $c > 0$ only depending on ϵ_0. Since the holomorphic function $f(\zeta_2,\zeta_3)$ defined in $\Omega^{\epsilon_0}_{a,\delta}$ is independent of ζ_1, f is the well-defined holomophic function in $\Omega^{\epsilon_0}_{a,\delta,\zeta_1}$. We want to show f is well-defined holomorphic function in $\Omega_{a,\delta,\zeta_1}$. Therefore, it is enough to show $\Omega_{a,\delta,\zeta_1} \subset \Omega^{\epsilon_0}_{a,\delta,\zeta_1}$ for the well-definedness of f in $\Omega_{a,\delta,\zeta_1}$. More specifically,

$$\Omega_{a,\delta,\zeta_1} \subset \Omega^{\epsilon_0}_{a,\delta,\zeta_1} \;\Leftrightarrow \rho(d\delta^{\frac{1}{\eta}}, \zeta'') - \rho(\zeta_1, \zeta'') < \epsilon_0 J_\delta(\zeta''),$$

where $\zeta'' = (\zeta_2,\zeta_3)$ and $|\zeta_1 - d\delta^{\frac{1}{\eta}}| < c\delta^{\frac{1}{\eta}}, |\zeta_2| < a$ and $|\zeta_3| < a$.

Proposition 3.6. *Given any small $\epsilon \le \epsilon_0$, there is a small $c > 0$ such that if $|\zeta_1 - d\delta^{\frac{1}{\eta}}| < c\delta^{\frac{1}{\eta}}, |\zeta_2| < a$ and $|\zeta_3| < a$, then*

$$|\rho(d\delta^{\frac{1}{\eta}}, \zeta'') - \rho(\zeta_1, \zeta'')| \lesssim \epsilon J_\delta(\zeta'').$$

Before proving Proposition 3.6, we note that from the standard interpolation method, we have the following fact: Let $(p_1, q_1), (p, q)$ and (p_2, q_2) be collinear points in the first quadrant of the plane, and $p_1 \le p \le p_2, q_2 \le q \le q_1$. Then, we have

$$|\zeta_1|^p |\zeta_2|^q \le |\zeta_1|^{p_1} |\zeta_2|^{q_1} + |\zeta_1|^{p_2} |\zeta_2|^{q_2}$$

for sufficiently small $\zeta_1, \zeta_2 \in \mathbb{C}$. In particular, this means that if $(\alpha, \beta) \in \Gamma_L$, then

$$|\zeta_1|^{\alpha_1+\beta_1} |\zeta_2|^{\alpha_2+\beta_2} \lesssim |\zeta_1|^{p_{\nu-1}} |\zeta_2|^{q_{\nu-1}} + |\zeta_1|^{p_\nu} |\zeta_2|^{q_\nu} \tag{3.20}$$

for some $\nu = 1, \cdots, N$.

Proof of Proposition 3.6. Define

$$J_\delta{}^\nu(\zeta'') = \delta + |\zeta_3| + \sum_{\nu=1}^{N} \delta^{\frac{p_\nu}{\eta}} |\zeta_2|^{q_\nu}.$$

In order to show the proposition, it is enough to show $J_\delta{}^\nu(\zeta'') \lesssim J_\delta(\zeta'')$ and $|\rho(d\delta^{\frac{1}{\eta}}, \zeta_2, \zeta_3) - \rho(\zeta_1, \zeta_2, \zeta_3)| \lesssim \epsilon J_\delta{}^\nu(\zeta'')$, where $|\zeta_1 - d\delta^{\frac{1}{\eta}}| < c\delta^{\frac{1}{\eta}}, |\zeta_2| < a$ and $|\zeta_3| < a$. By (3.8) and $a_{j,k}(\widetilde{e_\delta}) = j!k! \frac{\partial^{j+k}\rho}{\partial \zeta_2{}^j \partial \bar{\zeta_2}{}^k}(d\delta^{\frac{1}{\eta}}, 0, 0)$, we have

$$\left|\frac{\partial^{j+k}\rho}{\partial \zeta_2{}^j \partial \bar{\zeta_2}{}^k}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \lesssim |A_l(\widetilde{e_\delta})|$$

for $j + k = l$ with $l = 2, \cdots, m$. Therefore, Lemma 3.3 means that

$$\delta^{\frac{p_\nu}{\eta}} \approx \left|\frac{\partial^{q_\nu}\rho}{\partial \zeta_2{}^{\alpha_2^\nu} \partial \bar{\zeta}_2^{\beta_2^\nu}}(d\delta^{\frac{1}{\eta}}, 0, 0)\right| \lesssim |A_{q_\nu}(\widetilde{e_\delta})|,$$

where $\alpha_2^\nu + \beta_2^\nu = q_\nu, \alpha_2^\nu$ and $\beta_2^\nu > 0$. This shows $J_\delta{}^\nu(\zeta'') \lesssim J_\delta(\zeta'')$.

Let's estimate $|\rho(d\delta^{\frac{1}{\eta}}, \zeta'') - \rho(\zeta_1, \zeta'')|$. Let D_1 denote the differential operator either $\frac{\partial}{\partial \zeta_1}$ or $\frac{\partial}{\partial \bar{\zeta}_1}$. Then,

$$|\rho(\zeta_1, \zeta'') - \rho(d\delta^{\frac{1}{\eta}}, \zeta'')| \le c\delta^{\frac{1}{\eta}} \max_{|\zeta_1 - d\delta^{\frac{1}{\eta}}| < c\delta^{\frac{1}{\eta}}} |D_1\rho(\zeta_1, \zeta'')|. \tag{3.21}$$

Let's estimate $D_1\rho(\zeta_1, \zeta'')$. By (2.9), (3.18) and (3.19), we know

$$\rho(\zeta_1, \zeta'') = \mathrm{Re}(\Phi_3(\zeta)) + \sum_{\Gamma_L} a_{\alpha,\beta} \zeta_1{}^{\alpha_1} \bar{\zeta}_1^{\beta_1} \zeta_2{}^{\alpha_2} \bar{\zeta}_2^{\beta_2} + \mathcal{O}(|\Phi_3(\zeta)||(\zeta_1, \zeta_2, \Phi_3(\zeta))|$$
$$+ \sum_{\nu=1}^{N} \sum_{l=q_{\nu-1}}^{q_\nu} |\zeta_1|^{[t_l]+1} |\zeta_2|^l + |\zeta_2|^{m+1}).$$

Since $|\zeta_1 - d\delta^{\frac{1}{\eta}}| < c\delta^{\frac{1}{\eta}}$ and Φ_3 is independent of ζ_1, we have

$$|D_1\rho(\zeta_1, \zeta'')| \lesssim \sum_{\Gamma_L} \delta^{\frac{\alpha_1+\beta_1-1}{\eta}} |\zeta_2|^{\alpha_2+\beta_2} + |\Phi_3(\zeta)| + \sum_{\nu=1}^{N} \sum_{l=q_{\nu-1}}^{q_\nu} \delta^{\frac{[t_l]}{\eta}} |\zeta_2|^l. \tag{3.22}$$

Combining (3.21) with (3.22), we obtain

$$|\rho(\zeta_1, \zeta'') - \rho(d\delta^{\frac{1}{\eta}}, \zeta'')| \lesssim c\left(\sum_{\Gamma_L} \delta^{\frac{\alpha_1+\beta_1}{\eta}} |\zeta_2|^{\alpha_2+\beta_2} + |\Phi_3(\zeta)| + \sum_{\nu=1}^{N} \sum_{l=q_{\nu-1}}^{q_\nu} \delta^{\frac{[t_l]+1}{\eta}} |\zeta_2|^l\right)$$

With $\zeta_1 = d\delta^{\frac{1}{\eta}}$, (3.20) means $\sum_{\Gamma_L} \delta^{\frac{\alpha_1+\beta_1}{\eta}} |\zeta_2|^{\alpha_2+\beta_2} \lesssim J_\delta{}^\nu(\zeta'')$. Also, (3.7) and Lemma 3.3 gives $|\Phi_3(\zeta)| \lesssim |e_\delta| + |\zeta_3| + \sum_{l=1}^m |c_l(\widetilde{e_\delta})||\zeta_2|^l \lesssim \delta + |\zeta_3| + \sum_{l=1}^m \delta^{\frac{t_l}{\eta}} |\zeta_2|^l$. Since $(t_l, l) \in L_\nu$ for some $\nu = 1, \cdots, N$, again, (3.20) gives $|\Phi_3(\zeta)| \lesssim J_\delta{}^\nu(\zeta'')$. Furthermore, since $\delta^{\frac{[t_l]+1}{\eta}} |\zeta_2|^l \lesssim \delta^{\frac{t_l}{\eta}} |\zeta_2|^l$, the same argument as before gives $\sum_{\nu=1}^N \sum_{l=q_{\nu-1}}^{q_\nu} \delta^{\frac{[t_l]+1}{\eta}} |\zeta_2|^l \lesssim J_\delta{}^\nu(\zeta'')$. □

Now, we know that there is a holomorphic function $f(\zeta_1, \zeta_2, \zeta_3) = f(\zeta_2, \zeta_3)$ defined on $\Omega^{\epsilon_0}_{a,\delta,\zeta_1}$ such that

i) $\Omega_{a,\delta,\zeta_1} \subset \Omega^{\epsilon_0}_{a,\delta,\zeta_1}$

ii) $\left|\frac{\partial f}{\partial \zeta_3}(0, -\frac{b\delta)}{2}\right| \geq \frac{1}{2\delta}$ for a small constant $b > 0$.

Without loss of generality, we can assume $\Omega_{a,\delta,\zeta_1} \subset \Omega^{\frac{\epsilon_0}{2}}_{a,\delta,\zeta_1} \subset \Omega^{\epsilon_0}_{a,\delta,\zeta}$. For the boundedness of f in $\Omega^{\frac{\epsilon_0}{2}}_{\frac{a}{2},\delta,\zeta_1}$, we follow the same argument as Chapter 7 (p 462) in [4]. Before showing the boundedness, we define a polydisc $P_{a_1}(\zeta''_0)$ by

$$P_{a_1}(\zeta''_0) = \{\zeta'' = (\zeta_2, \zeta_3); |\zeta_2 - \zeta_2^0| < \tau(\widetilde{e_\delta}, a_1 J_\delta(\zeta''_0)) \text{ and } |\zeta_3 - \zeta_3^0| < a_1 J_\delta(\zeta''_0)\},$$

where $\zeta''_0 = (\zeta_2^0, \zeta_3^0)$ and $a_1 > 0$.

Theorem 3.7. *f is bounded holomorphic function in $\Omega^{\frac{\epsilon_0}{2}}_{\frac{a}{2},\delta,\zeta_1}$ such that*

$$\left|\frac{\partial f}{\partial \zeta_3}\left(0, -\frac{b\delta}{2}\right)\right| \geq \frac{1}{2\delta} \textit{ for a small constant } b > 0. \tag{3.23}$$

Proof. Since f is a L^2 holomorphic function in $\Omega^{\epsilon_0}_{a,\delta,\zeta_1}$ with (3.23), it is enough to show f is bounded in $\Omega^{\frac{\epsilon_0}{2}}_{\frac{a}{2},\delta,\zeta_1}$. Let $(\zeta_2^0, \zeta_3^0) \in \{\rho(d\delta^{\frac{1}{\eta}}, \zeta'') = \frac{\epsilon_0}{2} J_\delta(\zeta''), |\zeta_2| < \frac{3a}{4}, |\zeta_3| < \frac{3a}{4}\} \subset \Omega^{\epsilon_0}_{a,\delta,\zeta}$. By the similar property as (iii) of Proposition 4.3 in [4], if $\zeta''_0 = (\zeta_2^0, \zeta_3^0) \in \{\rho(d\delta^{\frac{1}{\eta}}, \zeta'') = \frac{\epsilon_0}{2} J_\delta(\zeta''), |\zeta_2| < \frac{3a}{4}, |\zeta_3| < \frac{3a}{4}\}$, then

$$P_{a_1}(\zeta''_0) \subset \Omega^{\epsilon_0}_{a,\delta,\zeta},$$

for some small constant $a_1 > 0$. We can apply the same argument as Chapter 7 (p 462) in [4] to obtain $|f(\zeta_2^0, \zeta_3^0)| \lesssim 1$. For all others points on the boundary and interior of $\Omega^{\frac{\epsilon_0}{2}}_{\frac{a}{2},\delta,\zeta_1}$, we can choose the polydics with fixed radius which is contained in $\Omega^{\epsilon_0}_{a,\delta,\zeta_1}$ and apply the same argument as Chapter 7 in [4]. $\square$

4 Proof of Theorem 1.1

In this section, we prove our main theorem. Before proving the Theorem, let's recall the notations for Hölder norm and Hölder space. For $U \in \mathbb{C}^n$, we denote by $\|u\|_{L_\infty(U)}$ the essential supremum of $u \in L_\infty(U)$ in U. For a real $0 < \epsilon < 1$, set

$$\|u\|_{\Lambda^\epsilon(U)} = \|u\|_{L_\infty(U)} + \sup_{z,w\in U} \frac{|u(w) - u(z)|}{|w - z|^\epsilon},$$
$$\Lambda^\epsilon(U) = \{u : \|u\|_{\Lambda^\epsilon(U)} < \infty\}$$

In here, $\|u\|_{\Lambda^\epsilon(U)}$ denote the Hölder norm of order ϵ.

By theorem 2.1, we can assume $\Omega = \{z \in \mathbb{C}^3; r(z) < 0\}$ and restate Theorem 1.1:

Theorem 4.1. *Let $\Omega = \{r(z) < 0\}$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$, where r given by theorem 2.1. Furthermore, if there exists a neighborhood U of 0 so that for all $\alpha \in L^{0,1}_\infty(\Omega)$ with $\bar\partial\alpha = 0$, there is a $u \in \Lambda_\epsilon(U \cap \overline{\Omega})$ and $C > 0$ such that $\bar\partial u = \alpha$ and*

$$\|u\|_{\Lambda^\epsilon(U\cap\overline{\Omega})} \le C\|\alpha\|_{L_\infty(\Omega)}, \tag{4.1}$$

then $\epsilon \le \frac{1}{\eta}$.

Proof. Let us consider $U' = \{(\zeta_1, \zeta_2, \zeta_3); \Phi(\zeta_1, \zeta_2, \zeta_3) \in U\}$ and $\rho = r \circ \Phi$ as (3.18) and (3.19). Let's choose $\beta = \bar\partial(\phi(\frac{|\zeta_1 - d\delta^{\frac{1}{\eta}}|}{c\delta^{\frac{1}{\eta}}})\phi(\frac{|\zeta_2|}{a/2})\phi(\frac{|\zeta_3|}{a/2})f(\zeta_2, \zeta_3))$, where

$$\phi(t) = \begin{cases} 1 & , |t| \le \frac{1}{2} \\ 0 & , |t| \ge \frac{3}{4} \end{cases}$$

Note that f is the well-defined bounded holomorphic function in $\Omega^{\frac{\epsilon}{2}}_{\frac{a}{2},\delta,\zeta_1}$ by Theorem 3.7. If we define $\alpha = (\Phi^{-1})^*\beta$, then $\bar\partial(\Phi^* u) = \Phi^*\bar\partial u = \Phi^*\alpha = \beta$. Therefore, if we set $U_1 = \Phi^* u = u \circ \Phi$, (4.1) means

$$\|U_1\|_{\Lambda^\epsilon(U'\cap\overline{\Omega})} \le C\|\beta\|_{L_\infty} \tag{4.2}$$

In here, we note that the definition of β means

$$\|\beta\|_{L^\infty} \lesssim \delta^{-\frac{1}{\eta}} \tag{4.3}$$

Now, let $h(\zeta_1,\zeta_2,\zeta_3) = U_1(\zeta_1,\zeta_2,\zeta_3) - \phi(\frac{|\zeta_1 - d\delta^{\frac{1}{\eta}}|}{c\delta^{\frac{1}{\eta}}})\phi(\frac{|\zeta_2|}{a/2})\phi(\frac{|\zeta_3|}{a/2})f(\zeta_2,\zeta_3)$. Then $\bar\partial U_1 = \beta$ means h is holomorphic. Set $q_1^\delta(\theta) = (d\delta^{\frac{1}{\eta}} + \frac{4}{5}c\delta^{\frac{1}{\eta}}e^{i\theta}, 0, -\frac{b\delta}{2})$ and $q_2^\delta(\theta) = (d\delta^{\frac{1}{\eta}} + \frac{4}{5}c\delta^{\frac{1}{\eta}}e^{i\theta}, 0, -b\delta)$, where $\theta \in \mathbb{R}$. From now on, we estimate the lower bound and upper bound of the integral

$$H_\delta = \left|\frac{1}{2\pi}\int_0^{2\pi}[h(q_1^\delta(\theta)) - h(q_2^\delta(\theta))]d\theta\right|.$$

From the definition of ϕ, (4.2), and (4.3) we have

$$H_\delta = \left|\frac{1}{2\pi}\int_0^{2\pi}[U_1(q_1^\delta(\theta)) - U_1(q_2^\delta(\theta))]d\theta\right| \lesssim \delta^\epsilon\|\beta\|_{L^\infty} \lesssim \delta^{\epsilon-\frac{1}{\eta}} \tag{4.4}$$

On the other hand, for the lower bound estimate, we start with an estimate of the holomorphic function f with a large nontangential derivative we constructed in theorem 3.7. The Taylor's theorem of f in ζ_3 and Cauchy's estimate means

$$f(0,\zeta_3) = f(0,-\frac{b\delta}{2}) + \frac{\partial f}{\partial\zeta_3}(0,-\frac{b\delta}{2})(\zeta_3 + \frac{b\delta}{2}) + \mathcal{O}(|\zeta_3 + \frac{b\delta}{2}|^2).$$

Now, if we take $\zeta_3 = -b\delta$, we have

$$f(0,-b\delta) - f(0,-\frac{b\delta}{2}) = \frac{\partial f}{\partial\zeta_3}(0,-\frac{b\delta}{2})(-\frac{b\delta}{2}) + \mathcal{O}(\delta^2).$$

Since $|\frac{\partial f}{\partial z_3}(0,-\frac{b\delta}{2})| \geq \frac{1}{2\delta}$, we know

$$|f(0,-b\delta) - f(0,-\frac{b\delta}{2})| = \left|\frac{\partial f}{\partial\zeta_3}(0,-\frac{b\delta}{2})(-\frac{b\delta}{2}) + \mathcal{O}(\delta^2)\right| \gtrsim 1 \tag{4.5}$$

for all sufficiently small $\delta > 0$. Returning to the lower bound estimate of H_δ, the Mean Value Property, (4.2), (4.3), and (4.5) give

$$\begin{aligned}
H_\delta &= \left|\frac{1}{2\pi}\int_0^{2\pi}[h(q_1^\delta(\theta))) - h(q_2^\delta(\theta))]d\theta\right| = \left|h(d\delta^{\frac{1}{\eta}},0,-\frac{b\delta}{2}) - h(d\delta^{\frac{1}{\eta}},0,-b\delta))\right| \\
&= \left|U_1(d\delta^{\frac{1}{\eta}},0,-\frac{b\delta}{2}) - f(0,-\frac{b\delta}{2}) - U_1(d\delta^{\frac{1}{\eta}},0,-b\delta) + f(0,-b\delta)\right| \\
&\geq \left|f(0,-b\delta) - f(0,-\frac{b\delta}{2})| - |U_1(d\delta^{\frac{1}{\eta}},0,-\frac{b\delta}{2}) - U_1(d\delta^{\frac{1}{\eta}},0,-b\delta)\right| \\
&\gtrsim 1 - \delta^{\epsilon-\frac{1}{\eta}}
\end{aligned} \tag{4.6}$$

If we combine (4.4) with (4.6), we have

$$1 \lesssim \delta^{\epsilon-\frac{1}{\eta}}. \tag{4.7}$$

If we assume $\epsilon > \frac{1}{\eta}$ and $\delta \to 0$, (4.7) will be a contradiction. Therefore, $\epsilon \leq \frac{1}{\eta}$. $\square$

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